UF Reversible Computing Project Memo #M14

The Adiabatic Principle

A generalized derivation

Michael Frank
UF CISE Dept.
<mpf@cise.ufl.edu>

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In this research memo, we consider the origins of what we consider to be the general principle of adiabatic systems, namely that the total energy dissipation E_{diss} of an adiabatic process asymptotically scales down in proportion to the speed at which that process takes place (that is, in inverse proportion to the total time over which the process is carried out, $E_{diss} \asymp 1/t_{tot}$). We wish to do this in a very generalized way that can be applied to any sort of adiabatic physical mechanism, whether it be electronic, mechanical, quantum-mechanical, *etc.* This will help us to justify the sort of technology-independent models of adiabatic/reversible computing which much of our work centers on.

We guess that probably some similar derivation has been published long ago somewhere in the phyics literature, but we deemed it easier to reinvent this simple result than to locate its first appearance. However, eventually, we should find the appropriate credit. The idea for the form of this general analysis was inspired by a reading of the analysis of the special case of electrical resistance in the textbook *University Physics*, 6th ed., by Sears, Zemansky, and Young (§28-9, pp. 556-557).

As a general model, let us consider the physical system in question as being characterized at any moment by a generalized coordinate x giving the system's position in configuration space. For any system composed of a fixed number of particles, x might simply be the vector of all particles' x, y, z coordinates in 3-D space. In an electronic system, the "position" could characterize the voltage states of different circuit nodes. For quantum processes involving particle creation and destruction, x might characterize the occupancy numbers of various particle states. But the general conception is that x characterizes the instantaneous state of the system, in whatever representation framework is important.

Actually, in classical mechanics, spatial position alone does not completely describe a system; momentum coodinates are also necessary. Quantum mechanics loosens this restriction, however, since momentum can be represented as an emergent phenomenon, arising from the wavelength of particle wave-packets in a Schrödinger wavefunction that ranges over the system's *position* configuration space alone. In any event, explicit momentum coordinates could be included if needed, but we do not worry about doing so in the present analysis.

Now, consider a process in which a system migrates along some (possibly complexly-shaped) desired path P through its configuration space. Note that P is the *desired* path; the system's real path may merely closely approximate P, due to small unwanted interactions that disturb its trajectory. Let the total length of P (the path integral of segments ds) be ℓ. Suppose the system makes progress along the desired path at some roughly constant velocity magnitude v (despite any shifts in its direction of motion). Call this its *path velocity*. Then the total time for the trip is $t_{tot} = \ell/v$. Given an *effective mass* m of the system, its approximate kinetic energy along path P at any moment during the trip is $E_k = \frac{1}{2}mv^2$ (ignoring relativistic corrections).

Suppose, now, that the system is subject to very frequent small "frictional" interactions with its environment, and that the nature of these interactions is that each of them saps some fraction f of the system's kinetic energy along the path P (converting this energy to heat). For example, in a flow of current through a resistor, the drifting electrons, in their rapid thermal motions, frequently scatter off of atoms of the material. Each time this happens, some fraction of the electron's drift kinetic energy (its extra energy in the direction of current flow, which itself is a small fraction of the total drift kinetic energy of the entire current) is thermalized.

Similarly, for an object falling in a viscous fluid, each collision between the object and an atom of the fluid is an elastic collision which saps a tiny fraction of the object's kinetic energy. The lost kinetic energy is assumed, in our context, to then be replished by an external force, *i.e.*, some potential energy bias favoring the system's forward motion along the desired path. Let $r_{int} = 1/t_f$ be the average rate at which these interactions occur, where where t_f is the *mean free time* of the system between interactions.

A crucial assumption at this point is that r_{int} is *independent* of the system's path velocity, at least for small v. That is, the interactions occur at some base rate, regardless of whether the system is moving or not, and this rate not increased significantly so long as v is reasonably small. (For sufficiently high v, the rate of interactions may vary, but that is OK since we are only interested here in the behavior as $v \to 0$.)

This independence assumption is true in scenarios such as the following: (a) Electrons in a resistor current, where electron thermal velocity is much greater than drift velocity; electrons frequently encounter atoms even when drift velocity goes to zero. (b) Object falling slowly through a viscous fluid; object encounters atoms frequently due to thermal motions in fluid, even when object is falling very slowly.

Given r_{int}, the expected number of these interactions that take place during the entire process is $n_{int} = r_{int} \cdot t_{tot}$. Each time the interaction happens, an amount $E_{int} = f \cdot E_k$ of the system's kinetic energy is lost (but, we assume, replenished by some source so that velocity remains roughly constant). Therefore, the total loss over the entire process is $E_{diss} = n_{int} \cdot E_{int}$. Expanding the definitions, we see that

$$\begin{aligned} E_{diss} &= n_{int} \cdot E_{int} \\ &= r_{int}\, t_{tot} \cdot f\, E_k \\ &= t_f^{-1} t_{tot} \cdot f\, \frac{1}{2} m \left(\frac{\ell}{t_{tot}}\right)^2 \\ &= \frac{1}{2} \cdot \frac{f\, m\, \ell^2}{t_f\, t_{tot}}. \end{aligned}$$

Already, we can see that the energy dissipated is inversely proportional to the time t_{tot} for the whole process, the key adiabatic principle which we wished to derive. Note that if the number of dissipative events was independent of time, the energy dissipated would scale down with kinetic energy, as t_{tot}^{-2}, but the increasing number of dissipative events with time raises this to order t_{tot}^{-1}. Note also that the total dissipation is inversely proportional to the mean free time t_f as well, so if we can better isolate the system from interactions (*e.g.* mechanical motion in vacuum, or electrical current in superconductors), the dissipation will decrease, as expected.

This attains the basic goal of this memo, but let us now explore some other convenient ways to express this dissipation relation, so as to better understand it.

Given f and r_{int}, we can now characterize an *energy decay coefficient* $d_E = f \cdot r_{int}$ which characterizes how rapidly a system's kinetic energy tends to decay away over time. That is, if $e_{diss}(t)$ is the total energy lost to friction so far at time t, its time derivative $\dot{e}_{diss} = d_E E_k$ (where the decay is modeled as continuous due to the large number of small cumulative energy losses). Note that the units of d_E are $time^{-1}$, and so its inverse $t_d = d_E^{-1}$ is a *decay time constant.* If the kinetic energy lost to friction is not replenished, an initial kinetic energy of $E_{k0} = E_k(t = t_0)$ would decay exponentially for $t \geq t_0$ according to $E_k(t) = E_{k0} \cdot e^{-t/t_d}$, the result of solving the differential equation describing this decay process.

Anyway, expressed in terms of d_E, we have $E_{diss} = \frac{1}{2} d_E\, m\, x^2/t$.

The product of d_E and the effective system mass m we will call the *effective viscosity* $\eta = m \cdot d_E$ because it has the same physical units ($mass/time$) as ordinary fluid viscosity, and indeed plays the same role in this more general setting. The frictional effect of energy loss can be characterized by a force aligned opposite the path direction, with magnitude scaling in proportion to viscosity and velocity, just as frictional forces scale when dealing with fluid viscosity. With our definitions, the exact relation is $F_{fric} = \frac{1}{2}\eta v,$. In terms of F_{fric} as we have so defined it, the equation for E_{diss} then simplifies to:

$$E_{diss} = F_{fric} \cdot \ell,$$

i.e., total energy lost equals the frictional force times the displacement distance.

Note that in order for E_{diss} to truly represent the total energy dissipation of the process, a key assumption was that only *kinetic* energy (but not potential energy or rest mass-energy) was lost as the system moved along the desired trajectory. But more generally, a system may experience some constant minimum rate of loss of its *total* energy, not just kinetic energy, away from the desired trajectory, due to effects such as thermal disturbance of the trajectory, wavefunction spreading, or quantum tunneling (all of which can be seen to be different variations on one general phenomenon).

A later memo will discuss the implications of this other type of loss on the limits of the adiabatic principle.